Journal of Engineering Mathematics **33:** 47–60, 1998.

Diffraction of elastic waves by four rigid strips embedded in an infinite orthotropic medium

S.C. MANDAL*[1], S.C. PAL[2] and M.L. GHOSH[3]
[1] *Department of Mathematics, Indian Institute of Science, Bangalore, 560 012, India.*
[1] *Jawaharlal Nehru Centre for Advanced Scientific Research, Jakkur, Bangalore, 560 064, India.*
[2] *Computer Centre, University of North Bengal, Darjeeling, 734 430, India.*
[3] *Department of Mathematics, Univeristy of North Bengal, Darjeeling, 734 430, India.*

Received 30 November 1994; accepted in revised form 1 July 1997

Abstract. In the present paper, the elastodynamic response of four coplanar rigid strips embedded in an infinite orthotropic medium due to elastic waves incident normally on the strips is analyzed. The resulting mixed boundary-value problem is solved by an integral-equation method. The normal stress and the vertical displacement are derived in closed analytic form. Numerical values of stress-intensity factors at the edges of the strips and vertical displacements at point in the plane of the strips for several orthotropic materials are calculated and plotted graphically to show the effect of material orthotropy.

Key words: diffraction, elasticity, waves, strips, orthotropic.

1. Introduction

In recent years, the study of problems involving cracks or inclusions in composite and anisotropic materials has gained much importance. The problems of diffraction of elastic waves by cracks or inclusions have aroused attention in the field of fracture mechanics in view of their application in seismology and geophysics. Studies of a single Griffith crack as well as two parallel and coplanar Griffith cracks have been made by Mal [1], Jain and Kanwal [2] and Itou [3]. The corresponding problems of diffraction by a single and two parallel rigid strips have been solved by Wickham [4], Jain and Kanwal [5] and Mandal and Ghosh [6], respectively. In most of these cases the problems were solved by the integral-equation technique, but the solutions of interesting problems involving the scattering of elastic waves by more than two coplanar Griffith cracks or strips are still lacking. The problem involving a single Griffith crack in an orthotropic medium was investigated by Kassir and Bandyopadhya [7], Shindo *et al.* [8] and De and Patra [9]. Shindo *et al.* [10] have investigated the impact response of symmetric edge cracks in an orthotropic strip. Mandal and Ghosh [11] considered the problem of interaction of elastic waves with a periodic array of coplanar Griffith cracks in an orthotropic elastic medium. The problem of scattering of elastic waves by a circular crack in a transversely isotropic medium was investigated by Kundu and Bostrom [12].

Here we consider the two-dimensional problem of diffraction of elastic waves by four coplanar parallel rigid strips embedded in an infinite orthotropic medium. The five-part mixed boundary-value problem is reduced to the solution of a set of integral equations. Following the technique developed by Srivastava and Lowengrub [13], the integral equations are solved. The normal stress under the strips and displacements outside the strips are derived in closed analytical form. To display the influence of the material orthotropy, numerical values of

* Permanent Address: Department of Mathematics, Jadavpur University, Calcutta, 700 032, India.

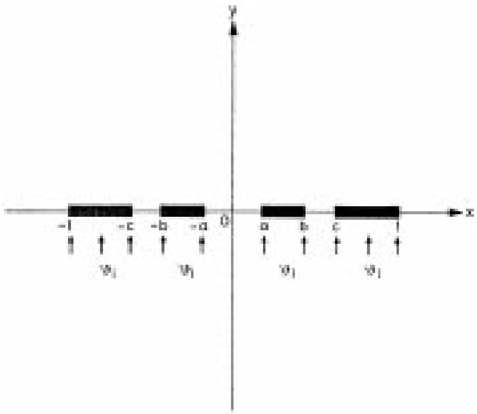

Figure 1. Geometry of the strips and incident field.

stress-intensity factors at the edges of the strips and vertical displacements are plotted against the dimensionless frequency and distance, respectively, for several orthotropic materials. This type of problem is important in view of their application in detecting the presence of inhomogeneities embedded in material structures and in seismology, while studying the scattering of elastic waves by inhomogeneities like rigid hard rock inside the earth.

2. Formulation of the problem

Consider the diifraction of a normally incident longitudinal wave by four coplanar and parallel rigid strips embedded in an infinite orthotropic elastic medium. The strips occupy the region $d_1 \leqslant |x_1| \leqslant d_2, d_3 \leqslant |x_1| \leqslant d$, $x_2 = 0, |x_3| < \infty$. Let E_i, μ_{ij} and $\nu_{ij} (i, j = 1, 2, 3)$ denote the engineering elastic constants of the material, where the subscripts 1, 2, 3 correspond to the x_1, x_2, x_3 directions which coincide with the axes of material orthotropy. Normalizing all lengths with respect to d and putting $x_1/d = x, x_2/d = y, x_3/d = z, d_1/d = a, d_2/d = b, d_3/d = c$, we define the rigid strips by $a \leqslant |x| \leqslant b, c \leqslant |x| \leqslant 1, y = 0, |z| < \infty$ (Figure 1).

Let a time-harmonic wave, given by $u_i = 0$ and $v_i = v_0$ exp$[i(ky - \omega t)]$, where $k = \omega d/c_s\sqrt{c_{22}}, c_s = (\mu_{12}/\rho)^{1/2}$ and v_0 is a constant, travelling in the direction of positive y-axis, be incident normally on the strips. The non-zero stress components τ_{yy} and τ_{xy} are given by

$$\tau_{yy}/\mu_{12} = c_{12}\frac{\partial u}{\partial x} + c_{22}\frac{\partial v}{\partial y}, \quad \tau_{xy}/\mu_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \tag{2.1}$$

where $c_{ij} (i, j = 1, 2)$ are nondimensional parameters related to the elastic constants by the relations

$$c_{11} = E_1/\mu_{12}(1 - \nu_{12}^2 E_2/E_1), \quad c_{22} = E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = c_{11}E_2/E_1 \tag{2.2}$$

$$c_{12} = \nu_{12}E_2/\mu_{12}(1 - \nu_{12}^2 E_2/E_1) = \nu_{12}c_{22} = \nu_{21}c_{11}.$$

The constants E_i and ν_{ij} satisfy the Maxwell relation $\nu_{ij}/E_i = \nu_{ji}/E_j$.

The equations of motion for orthotropic material in terms of the displacements are

$$c_{11}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (1+c_{12})\frac{\partial^2 v}{\partial x \partial y} = \frac{d^2}{c_s^2}\frac{\partial^2 u}{\partial t^2},$$
$$c_{22}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + (1+c_{12})\frac{\partial^2 u}{\partial x \partial y} = \frac{d^2}{c_s^2}\frac{\partial^2 v}{\partial t^2}, \tag{2.3}$$

where u, v are the displacement components of the scattered field. The boundary conditions are

(i) $u(x, y, t) = 0, v(x, y, t) + v_i(x, y, t) = 0$ across $y = 0$ on the surface of the strips;
(ii) u and v are continuous across $y = 0$ for $|x| < \infty$;
(iii) τ_{yy}, τ_{xy} are continuous across $y = 0$ outside the strips.

Further, the scattered field should satisfy the radiation condition at infinity.

Substituting $u(x, y, t) = u(x, y)$ exp$(-i\omega t)$ and $v(x, y, t) = v(x, y)\ exp(-i\omega t)$ we reduce our problem to the solution of the equations

$$c_{11}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (1+c_{12})\frac{\partial^2 v}{\partial x \partial y} + \frac{d^2\omega^2}{c_s^2}u = 0,$$
$$c_{22}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + (1+c_{12})\frac{\partial^2 u}{\partial x \partial y} + \frac{d^2\omega^2}{c_s^2}v = 0. \tag{2.4}$$

Boundary conditions on u and v suggest that u and v are odd and even functions of y, respectively. Accordingly, Equations (2.4) are to be solved subject to the boundary conditions

$$v(x, 0) = -v_0, \quad x \in I_2, I_4 \tag{2.5}$$

$$\tau_{yy}(x, 0) = 0, \quad x \in I_1, I_3, I_5 \tag{2.6}$$

$$u(x, 0) = 0, \quad |x| < \infty \tag{2.7}$$

with $I_1 = (0, a), I_2 = (a, b), I_3 = (b, c), I_4 = (c, 1), I_5 = (1, \infty)$.

Henecforth the time factor exp$(-i\omega t)$ which is common to all field variables would be omitted in the sequel.

The solutions of Equations (2.4) are taken as

$$u(x, y) = \pm\frac{2}{\pi}\int_0^\infty [A_1(\xi)\ \exp(-\gamma_1|y|) + A_2(\xi)\ \exp(-\gamma_2|y|)]\ \sin\ \xi x\, \mathrm{d}\xi, y \gtrless 0 \tag{2.8}$$

$$v(x, y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1 A_1(\xi)\ \exp(-\gamma_1|y|) + \alpha_2 A_2(\xi)\ \exp(-\gamma_2|y|)]\ \cos\ \xi x\, \mathrm{d}\xi \tag{2.9}$$

where

$$\alpha_i = \frac{c_{11}\xi^2 - k_s^2 - \gamma_i^2}{(1+c_{12})\gamma_i}, \quad i = 1, 2, \quad k_s^2 = \frac{d^2\omega^2}{c_s^2} \tag{2.10}$$

and $A_i(\xi)(i = 1, 2)$ are the unknowns to be solved, γ_1^2, γ_2^2 are the roots of the equation

$$c_{22}\gamma^4 + \{(c_{12}^2 + 2c_{12} - c_{11}c_{22})\xi^2 + (1+c_{22})k_s^2\}\gamma^2 + (c_{11}\xi^2 - k_s^2)(\xi^2 - k_s^2) = 0. \tag{2.11}$$

From the boundary condition (2.7) it is found that

$$A_2(\xi) = -A_1(\xi).$$

Therefore, displacements u, v and stresses τ_{yy}, τ_{xy} finally can be written as

$$u(x,y) = \frac{2}{\pi}\int_0^\infty [\exp(-\gamma_1|y|) - \exp(-\gamma_2|y|)]A_1(\xi)\,\sin\,\xi x\,\mathrm{d}\xi, y > 0, \tag{2.12}$$

$$v(x,y) = \frac{2}{\pi}\int_0^\infty \frac{1}{\xi}[\alpha_1\,\exp(-\gamma_1|y|) - \alpha_2\,\exp(-\gamma_2|y|)]A_1(\xi)\,\cos\,\xi x\,\mathrm{d}\xi, \tag{2.13}$$

$$\begin{aligned}\tau_{yy}/\mu_{12} \;=\; & \frac{2}{\pi}\int_0^\infty \left[\left(c_{12}\xi - \frac{c_{22}\alpha_1\gamma_1}{\xi}\right)\,\exp(-\gamma_1|y|) - \right. \\ & \left. - \left(c_{12}\xi - \frac{c_{22}\alpha_2\gamma_2}{\xi}\right)\,\exp(-\gamma_2|y|)\right] A_1(\xi)\,\cos\,\xi x\,\mathrm{d}\xi, y > 0,\end{aligned} \tag{2.14}$$

$$\begin{aligned}\tau_{xy}/\mu_{12} \;=\; & -\frac{2}{\pi}\int_0^\infty \left[(\gamma_1 + \alpha_1)\,\exp(-\gamma_1|y|) - (\gamma_2 + \alpha_2)\,\exp(-\gamma_2|y|)\right] A_1(\xi)\,\times \\ & \times\,\sin\,\xi x\,\mathrm{d}\xi.\end{aligned} \tag{2.15}$$

Next, putting $A(\xi) = \frac{\alpha_1\gamma_1 - \alpha_2\gamma_2}{\xi}A_1(\xi)$, we see that the boundary conditions (2.5) and (2.6) lead to the following integral equations in $A(\xi)$:

$$\int_0^\infty \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2}\right) A(\xi)\,\cos\,\xi x\,\mathrm{d}\xi = -\frac{\pi}{2}v_0, \quad x \in I_2, I_4 \tag{2.16}$$

and

$$\int_0^\infty A(\xi)\,\cos\,\xi x\,\mathrm{d}\xi = 0, \quad x \in I_1, I_3, I_5. \tag{2.17}$$

3. Solution of the problem

We consider the solution of the integral equations (2.16) and (2.17) in the form

$$A(\xi) = \int_a^b t f(t^2)\,\cos\,\xi t\,\mathrm{d}t + \int_c^1 u g(u^2)\cos\,\xi u\,\mathrm{d}u, \tag{3.1}$$

where $f(t^2)$ and $g(u^2)$ are unknown functions to be determined.

By the choice of $A(\xi)$ given by (3.1), the relation (2.17) is satisfied automatically and Equation (2.16) becomes

$$\begin{aligned}&\int_a^b t f(t^2)\,\mathrm{d}t \int_0^\infty \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2}\right)\,\cos\,\xi x\,\cos\,\xi t\,\mathrm{d}\xi + \\ &\quad + \int_c^1 u g(u^2)\mathrm{d}u \int_0^\infty \left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2}\right)\cos\,\xi x\,\cos\,\xi u\,\mathrm{d}\xi = -\frac{\pi}{2}v_0, \quad x \in I_2, I_4.\end{aligned} \tag{3.2}$$

Using the relation

$$\frac{\sin\,\xi x\,\sin\,\xi t}{\xi^2} = \int_0^x \int_c^t \frac{wvJ_0(\xi w)J_0(\xi v)\mathrm{d}v\,\mathrm{d}w}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}},$$

we may convert the above equation to the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b tf(t^2)\,\mathrm{d}t\frac{\partial}{\partial t}\int_0^x\int_c^t \frac{vwL_1(v,w)\mathrm{d}w\,\mathrm{d}v}{(x^2-w^2)^{1/2}(t^2-v^2)^{1/2}}+$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_c^1 ug(u^2)\mathrm{d}u\frac{\partial}{\partial u}\int_0^x\int_0^u \frac{vwL_1(v,w)\mathrm{d}w\,\mathrm{d}v}{(x^2-w^2)^{1/2}(u^2-v^2)^{1/2}} = -\frac{\pi}{2}v_0, \quad x\in I_2, I_4, \tag{3.3}$$

where

$$L_1(v,w) = \int_0^\infty \left(\frac{\alpha_1-\alpha_2}{\alpha_1\gamma_1-\alpha_2\gamma_2}\right) J_0(\xi w)J_0(\xi v)\mathrm{d}\xi. \tag{3.4}$$

By a contour integration technique [11] the infinite integral in $L_1(v,w)$ can be converted to the following finite integrals

$$\begin{aligned} L_1(v,w) \;=\; -i\Bigg[&\int_0^{1/\sqrt{c_{11}}} \frac{c_{11}\eta^2-1-\bar\gamma_1\bar\gamma_2}{\bar\gamma_1\bar\gamma_2(\bar\gamma_1+\bar\gamma_2)}J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,\mathrm{d}\eta - \\ &-\int_{1/\sqrt{c_{11}}}^1 \frac{c_{11}\eta^2-1+\bar\gamma_2'^2}{\bar\gamma_2'(\bar\gamma_2'^2+\bar\gamma_2'^2)}J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,\mathrm{d}\eta\Bigg], w>v \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} &\bar\gamma_1 = [\tfrac12\{R_1-(R_1^2-4R_2)^{1/2}\}]^{1/2}, \quad \bar\gamma_2 = [\tfrac12\{R_1+(R_1^2-4R_2)^{1/2}\}]^{1/2}, \\ &\bar\gamma_1' = [\tfrac12\{-R_1+(R_1^2-4R_3)^{1/2}\}]^{1/2}, \quad \bar\gamma_2' = [\tfrac12\{R_1+(R_1^2+4R_3)^{1/2}\}]^{1/2}, \\ &R_1 = \frac{1}{c_{22}}\{(c_{12}^2+2c_{12}-c_{11}c_{22})\eta^2+(1+c_{22})\}, \\ &R_2 = \frac{c_{11}}{c_{22}}(1-\eta^2)\left(\frac{1}{c_{11}}-\eta^2\right), \quad R_3 = \frac{c_{11}}{c_{22}}(1-\eta^2)\left(\eta^2-\frac{1}{c_{11}}\right). \end{aligned} \tag{3.6}$$

The corresponding expression of $L_1(v,w)$ for $w<v$ follows from (3.5) by the interchanging of w and v.

Substituting the series expansion of $J_0(\;)$ and $H_0^{(1)}(\;)$ for small k_s in (3.5) we find, after some algebraic manipulation,

$$\begin{aligned} L_1(v,w) \;&=\; \frac{2}{\pi}\Bigg[(\gamma+\log(k_s w/2)-\tfrac12\pi i)M+N \\ &\qquad -\frac{(w^2+v^2)}{4}Rk_s^2\,\log\,k_s\Bigg]+O(k_s^2), \quad w>v \\ &=\; \frac{2}{\pi}\Bigg[(\gamma+\log(k_s v/2)-\tfrac12\pi i)M+N \\ &\qquad -\frac{(w^2+v^2)}{4}Rk_s^2\,\log\,k_s\Bigg]+O(k_s^2), \quad v>w \end{aligned} \tag{3.7}$$

where $\gamma = 0{\cdot}5772157...$ is Euler's constant,

$$M = \int_0^{1/\sqrt{c_{11}}} \frac{c_{11}\eta^2 - 1 - \bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1 + \bar{\gamma}_2)}\,\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_2'^2}{\bar{\gamma}_2'(\bar{\gamma}_1'^2 + \bar{\gamma}_2'^2)}\,\mathrm{d}\eta, \tag{3.8}$$

$$N = \int_0^{1/\sqrt{c_{11}}} \frac{c_{11}\eta^2 - 1 - \bar{\gamma}_1\bar{\gamma}_2}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1 + \bar{\gamma}_2)}\,\log\,\eta\,\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{c_{11}\eta^2 - 1 + \bar{\gamma}_2'^2}{\bar{\gamma}_2'(\bar{\gamma}_1'^2 + \bar{\gamma}_2'^2)}\,\log\,\eta\,\mathrm{d}\eta, \tag{3.9}$$

and

$$R = \int_0^{1/\sqrt{c_{11}}} \frac{\eta^2(c_{11}\eta^2 - 1 - \bar{\gamma}_1\bar{\gamma}_2)}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1 + \bar{\gamma}_2)}\,\mathrm{d}\eta - \int_{1/\sqrt{c_{11}}}^{1} \frac{\eta^2(c_{11}\eta^2 - 1 + \bar{\gamma}_2'^2)}{\bar{\gamma}_2'(\bar{\gamma}_1'^2 + \bar{\gamma}_2'^2)}\,\mathrm{d}\eta. \tag{3.10}$$

Now, differentiating both sides of the relation (3.2) with respect to x, we obtain

$$\int_a^b tf(t^2)\,\mathrm{d}t \int_0^\infty \xi\left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2}\right)\,\sin\,\xi x\,\cos\,\xi t\,\mathrm{d}\xi +$$
$$+ \int_c^1 ug(u^2)\,\mathrm{d}u \int_0^\infty \xi\left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2}\right)\,\sin\,\xi x\,\cos\,\xi u\,\mathrm{d}\xi = 0, \quad x \in I_2, I_4$$

Following a similar procedure as was used for the derivation of Equation (3.3), we obtain

$$x\int_a^b \frac{tf(t^2)}{x^2 - t^2}\,\mathrm{d}t + x\int_c^1 \frac{ug(u^2)}{x^2 - u^2}\,\mathrm{d}u = \int_a^b tf(t^2)\,\mathrm{d}t\frac{\partial}{\partial t}\int_0^x\int_c^t \frac{vwL_2(v,w)\,\mathrm{d}w\,\mathrm{d}v}{(x^2 - w^2)^{1/2}(t^2 - v^2)^{1/2}}$$
$$+ \int_c^1 ug(u^2)\,\mathrm{d}u\frac{\partial}{\partial u}\int_0^x\int_0^u \frac{vwL_2(v,w)\,\mathrm{d}w\,\mathrm{d}v}{(x^2 - w^2)^{1/2}(u^2 - v^2)^{1/2}}, \quad x \in I_2, I_4, \tag{3.11}$$

where

$$L_2(v,w) = \int_0^\infty \left[\xi - \frac{\xi^2}{\theta}\left(\frac{\alpha_1 - \alpha_2}{\alpha_1\gamma_1 - \alpha_2\gamma_2}\right)\right] J_0(\xi w)J_0(\xi v)\,\mathrm{d}\xi, \tag{3.12}$$

$$\theta = \frac{c_{11} + N_1N_2}{N_1 + N_2}, \tag{3.13}$$

$$N_1^2 = \frac{1}{2c_{22}}\left[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) + \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}}\right],$$

and

$$N_2^2 = \frac{1}{2c_{22}}\left[-(c_{12}^2 + 2c_{12} - c_{11}c_{22}) - \sqrt{(c_{12}^2 + 2c_{12} - c_{11}c_{22})^2 - 4c_{11}c_{22}}\right]. \tag{3.14}$$

We use contour integration technique mentioned earlier and get from (3.12)

$$L_2(v,w) = \frac{ik_s^2}{\theta}\left[\int_0^{1/\sqrt{c_{11}}} \frac{\eta^2(c_{11}\eta^2 - 1 - \bar{\gamma}_1\bar{\gamma}_2)}{\bar{\gamma}_1\bar{\gamma}_2(\bar{\gamma}_1 + \bar{\gamma}_2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,\mathrm{d}\eta -\right.$$
$$\left. - \int_{1/\sqrt{c_{11}}}^{1} \frac{\eta^2(c_{11}\eta^2 - 1 + \bar{\gamma}_2'^2)}{\bar{\gamma}_2'(\bar{\gamma}_1'^2 + \bar{\gamma}_2'^2)} J_0(k_s\eta v)H_0^{(1)}(k_s\eta w)\,\mathrm{d}\eta\right], w > v \tag{3.15}$$

By a process similar to that which led to Equation (3.7), Equation (3.15) for small values of k_s can be written as

$$L_2(v,w) = -\frac{2}{\pi} P k_s^2 \log k_s + O(k_s^2), \tag{3.16}$$

where $P = \frac{1}{\theta} R$ and R is given by (3.10).

Now, let us consider

$$f(t^2) = f_0(t^2) + k_s^2 \log k_s f_1(t^2) + O(k_s^2)$$

and

$$g(u^2) = g_0(u^2) + k_s^2 \log k_s g_1(u^2) + O(k_s^2). \tag{3.17}$$

Putting the above expansions of $f(t^2), g(u^2)$ and the value of $L_2(v,w)$ given by (3.16) in Equation (3.11) and equating the coefficients of like powers of k_s, we obtain,

$$\int_a^b \frac{t f_0(t^2)}{x^2 - t^2}\,\mathrm{d}t + \int_c^1 \frac{u g_0(u^2)}{x^2 - u^2}\,\mathrm{d}u = 0, \quad x \in I_2, I_4 \tag{3.18}$$

and

$$\begin{aligned}&\int_a^b \frac{t f_1(t^2)}{x^2 - t^2}\,\mathrm{d}t + \int_c^1 \frac{u g_1(u^2)}{x^2 - u^2}\,\mathrm{d}u \\ &\quad = -\frac{2P}{\pi}\left[\int_a^b t f_0(t^2)\mathrm{d}t + \int_0^1 u g_0(u^2)\,\mathrm{d}u\right], x \in I_2, I_4.\end{aligned} \tag{3.19}$$

Following Srivastava and Lowengrub [13], we may obtain the solutions of the above integral equation (3.18) as

$$\begin{aligned}f_0(t^2) \;=\;& D_1 \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2} \left(\frac{c^2-t^2}{1-t^2}\right)^{1/2} \frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}} - \\ & -D_2 \left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2} \frac{1}{\sqrt{(1-t^2)(c^2-t^2)}}, \quad x \in I_2\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}g_0(u^2) \;=\;& D_1 \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2} \left(\frac{u^2-c^2}{1-u^2}\right)^{1/2} \frac{1}{\sqrt{(u^2-a^2)(u^2-b^2)}} + \\ & +D_2 \left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2} \frac{1}{\sqrt{(1-u^2)(u^2-c^2)}}, \quad x \in I_4,\end{aligned} \tag{3.21}$$

where D_1 and D_2 are constants which can be calculated as follows.

We substitute the value of $L_1(v,w)$ from (3.7) as well as the expansion of $f(t^2)$ and $g(u^2)$ as obtained from (3.17), (3.20) and (3.21) up to $O(k_s^2\ \log\ k_s)$ in Equation (3.3). When the coefficients of like powers of k_s from both sides of the resulting equation are equated, we get, after some manipulation, the following results:

$$D_1 = -v_0\frac{\pi^2}{4}\frac{(X_4-X_2)}{(X_1X_4-X_2X_3)}; \qquad D_2 = -v_0\frac{\pi^2}{4}\frac{(X_3-X_1)}{X_2X_3-X_1X_4)}, \tag{3.22}$$

where

$$X_1 = \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left[\left\{\left(\gamma+\log(k_s/2)-\frac{\pi i}{2}\right)M+N\right\}(J_1+J_3) + \tfrac{1}{2}MJ_1\ \log(b^2-a^2)+MJ_5\right], \tag{3.23}$$

$$X_2 = \left\{\left(\gamma+\log(k_s/2)-\frac{\pi i}{2}\right)M+N\right\}(J_4-J_2) - \tfrac{1}{2}MJ_2\ \log(b^2-a^2)+MJ_6, \tag{3.24}$$

$$X_3 = \left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left[\left\{\left(\gamma+\log(k_s/2)-\frac{\pi i}{2}\right)M+N\right\}(J_1+J_3) + \tfrac{1}{2}MJ_3\ \log(1-c^2)+MJ_7\right], \tag{3.25}$$

$$X_4 = \left\{\left(\gamma+\log(k_s/2)-\frac{\pi i}{2}\right)M+N\right\}(J_4-J_2)+\tfrac{1}{2}MJ_4\ \log(1-c^2)-MJ_8, \tag{3.26}$$

$$J_1 = \int_a^b\left(\frac{c^2-t^2}{1-t^2}\right)^{1/2}\frac{t\,\mathrm{d}t}{\sqrt{(t^2-a^2)(b^2-t^2)}},$$

$$J_2 = \int_a^b\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\frac{t\,\mathrm{d}t}{\sqrt{(1-t^2)(c^2-t^2)}},$$

$$J_3 = \int_c^1\left(\frac{u^2-c^2}{1-u^2}\right)^{1/2}\frac{u\,\mathrm{d}u}{\sqrt{(u^2-a^2)(u^2-b^2)}},$$

$$J_4 = \int_c^1\left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2}\frac{u\,\mathrm{d}u}{\sqrt{(1-u^2)(u^2-c^2)}},$$

$$J_5 = \int_c^1\left(\frac{u^2-c^2}{1-u^2}\right)^{1/2}\frac{u\ \log(\sqrt{u^2-b^2}+\sqrt{u^2-a^2})}{\sqrt{(u^2-a^2)(u^2-b^2)}}\,\mathrm{d}u,$$

$$J_6=\int_c^1\left(\frac{u^2-a^2}{u^2-b^2}\right)^{1/2}\frac{u\,\log(\sqrt{u^2-b^2}+\sqrt{u^2-a^2})}{\sqrt{(1-u^2)(u^2-c^2)}}\,\mathrm{d}u,$$

$$J_7=\int_a^b\left(\frac{c^2-t^2}{1-t^2}\right)^{1/2}\frac{t\,\log(\sqrt{c^2-t^2}+\sqrt{1-t^2})}{\sqrt{(t^2-a^2)(b^2-t^2)}}\,\mathrm{d}t,$$

$$J_8=\int_a^b\left(\frac{t^2-a^2}{b^2-t^2}\right)^{1/2}\frac{t\,\log(\sqrt{c^2-t^2}+\sqrt{1-t^2})}{\sqrt{(1-t^2)(c^2-t^2)}}\,\mathrm{d}t.$$

4. Stress intensity factors and displacement

The normal stress $\tau_{yy}(x,y)$ on the plane $y=0$ can be found from the relations (2.14), (3.1), (3.17), (3.20) and (3.21) as

$$\begin{aligned}\tau_{yy}(x,0) &= -\frac{\mu_{12}c_{22}x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\left\{D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left(\frac{c^2-x^2}{1-x^2}\right)^{1/2}-\right.\\ &\quad\left.-\frac{D_2(x^2-a^2)}{\sqrt{(1-x^2)(c^2-x^2)}}\right\}+O(k_s^2\,\log\,k_s),\quad x\in I_2\\ &= -\frac{\mu_{12}c_{22}x}{\sqrt{(x^2-c^2)(1-x^2)}}\left\{D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\frac{(x^2-c^2)}{\sqrt{(x^2-a^2)(x^2-b^2)}}+\right.\\ &\quad\left.+D_2\left(\frac{x^2-a^2}{x^2-b^2}\right)^{1/2}\right\}+O(k_s^2\,\log\,k_s),\quad x\in I_4.\end{aligned}\tag{4.1}$$

Defining the stress intensity factors at the edges of the strips by the relations

$$K_a=\underset{x\to a+}{Lt}\left|\frac{\tau_{yy}(x,0)\sqrt{(x-a)}}{v_0\mu_{12}}\right|,\quad K_b=\underset{x\to b-}{Lt}\left|\frac{\tau_{yy}(x,0)\sqrt{(b-x)}}{v_0\mu_{12}}\right|,$$

$$K_c=\underset{x\to c+}{Lt}\left|\frac{\tau_{yy}(x,0)\sqrt{(x-c)}}{v_0\mu_{12}}\right|,\quad K_1=\underset{x\to 1-}{Lt}\left|\frac{\tau_{yy}(x,0)\sqrt{(1-x)}}{v_0\mu_{12}}\right|,$$

we get

$$K_a=\left|\frac{c_{22}\sqrt{a}D_1}{\sqrt{2(b^2-a^2)}}\right|,\tag{4.2}$$

$$K_b=\left|\frac{c_{22}\sqrt{b}}{\sqrt{2(b^2-a^2)}}\left\{D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}\left(\frac{c^2-b^2}{1-b^2}\right)^{1/2}-\frac{D_2(b^2-a^2)}{\sqrt{(1-b^2)(c^2-b^2)}}\right\}\right|,\tag{4.3}$$

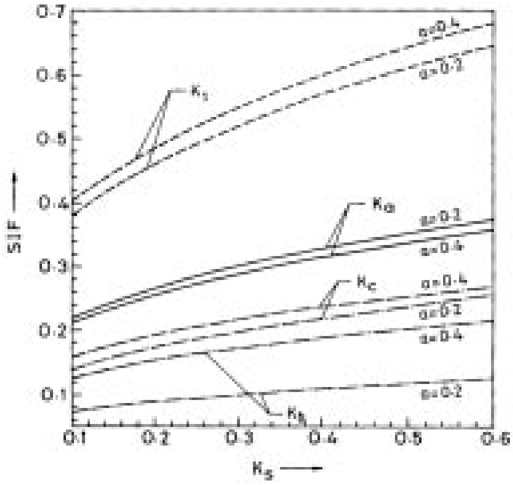

Figure 2. Stress intensity factors vs. frequency k_s for generalized plane stress (for material of type III).

$$K_c = \left| \frac{c_{22}\sqrt{c}}{\sqrt{2(1-c^2)}} D_2 \left(\frac{c^2 - a^2}{c^2 - b^2} \right)^{1/2} \right|, \tag{4.4}$$

$$K_1 = \left| \frac{c_{22}}{\sqrt{2(1-c^2)}} \left\{ \frac{D_1(1-c^2)}{\sqrt{(1-b^2)(c^2-a^2)}} + D_2 \left(\frac{1-a^2}{1-b^2} \right)^{1/2} \right\} \right|. \tag{4.5}$$

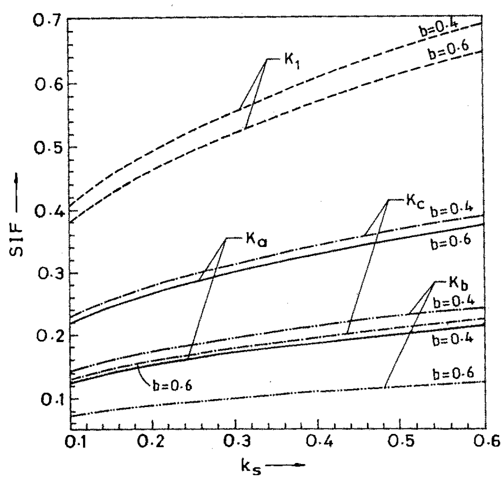

Figure 3. Stress intensity factors vs. frequency k_s for generalized plane stress (for material of type III).

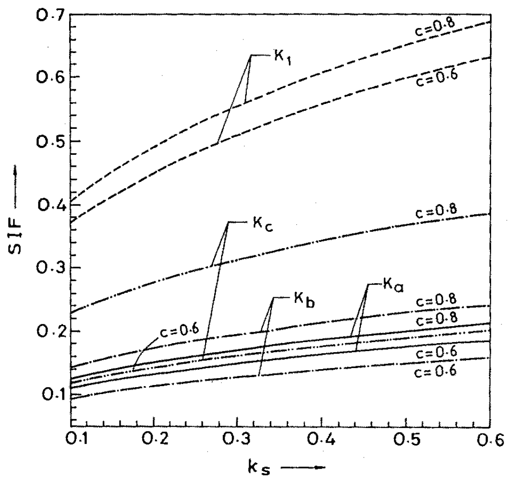

Figure 4. Stress intensity factors vs. frequency k_s for generalized plane stress (for material of type III).

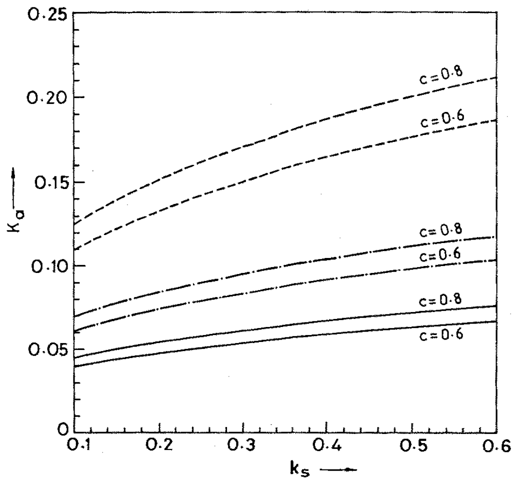

Figure 5. Stress intensity factor K_a vs. frequency k_s for generalized plane stress. (—— Type I, –.–.– Type II, - - - - - Type III).

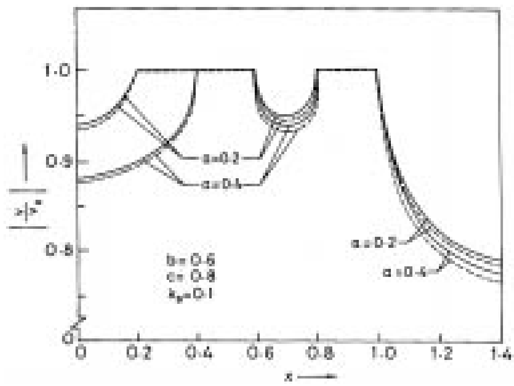

Figure 6. Vertical displacement $|v/v_0|$ vs. distance x for generalized plane stress. (—— Type I, - - - - - Type II).

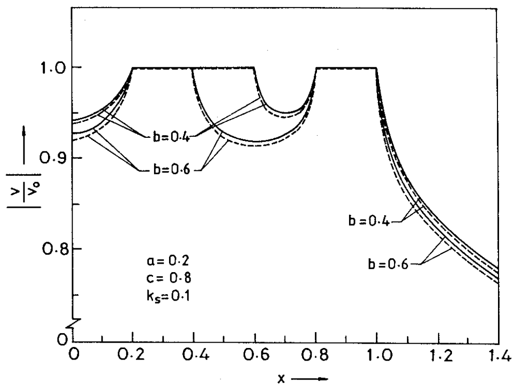

Figure 7. Vertical displacement $|v/v_0|$ vs. distance x for generalized plane stress. (—— Type I, - - - - - Type II).

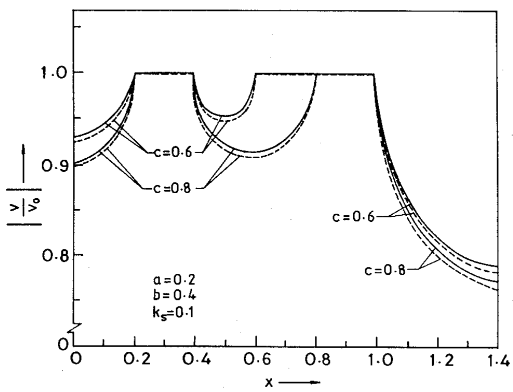

Figure 8. Vertical displacement $|v/v_0|$ vs. distance x for generalized plane stress. (—— Type I, - - - - - Type II).

The vertical displacement $v(x,y)$ on the plane $y=0$ can be obtained from Equations (2.13), (3.1), (3.17), (3.20) and (3.21) as

$$\begin{aligned} v(x,0) \;=\; \frac{4}{\pi^2}\Bigg[&\left\{\left(\gamma+\log(k_s)-\frac{\pi i}{2}\right)M+N\right\} \\ &\times\left\{D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}(J_1+J_3)+D_2(J_4-J_2)\right\} \\ &+\frac{M}{2}\left\{D_1\left(\frac{1-a^2}{c^2-a^2}\right)^{1/2}(J_9+J_{11})+D_2(J_{12}-J_{10})\right\}\Bigg], \end{aligned}$$

$$x \in I_1, I_3, I_5 \tag{4.6}$$

where

$$J_9 = \int_a^b \left(\frac{c^2 - t^2}{1 - t^2}\right)^{1/2} \frac{t \log|t^2 - x^2|}{\sqrt{(t^2 - a^2)(b^2 - t^2)}}\,\mathrm{d}t,$$

$$J_{10} = \int_a^b \left(\frac{t^2 - a^2}{b^2 - t^2}\right)^{1/2} \frac{t \log|t^2 - x^2|}{\sqrt{(1 - t^2)(c^2 - t^2)}}\,\mathrm{d}t,$$

$$J_{11} = \int_c^1 \left(\frac{u^2 - c^2}{1 - u^2}\right)^{1/2} \frac{u \log|u^2 - x^2|}{\sqrt{(u^2 - a^2)(u^2 - b^2)}}\,\mathrm{d}u,$$

$$J_{12} = \int_c^1 \left(\frac{u^2 - a^2}{u^2 - b^2}\right)^{1/2} \frac{u \log|u^2 - x^2|}{\sqrt{(u^2 - c^2)(1 - u^2)}}\,\mathrm{d}u.$$

In order to obtain the solution of the problem corresponding to two rigid strips, taking $b \to c$, we find from (3.20) and (3.21) that in this particular case

$$\begin{aligned} f_0(t^2) \;=\; g_0(t^2) &= D_1 \left(\frac{1 - a^2}{c^2 - a^2}\right)^{1/2} \frac{1}{\sqrt{(t^2 - a^2)(1 - t^2)}} \\ &\quad - D_2 \left(\frac{t^2 - a^2}{1 - t^2}\right)^{1/2} \frac{1}{(b^2 - t^2)}, \quad a \leqslant t \leqslant 1. \end{aligned}$$

It can further be shown that $X_1 = X_3$, so that

$$D_2 = 0, \quad \text{and} \quad D_1 = -\frac{V_0 \pi^2}{4X_1},$$

where

$$X_1 = \frac{\pi}{2} \left(\frac{1 - a^2}{c^2 - a^2}\right)^{1/2} [\{\gamma + \log(k_2/2) - \tfrac{1}{2}\pi i + \log(1 - a^2)^{1/2}\}M + N].$$

It can easily be shown that in the isotropic case this result is identical with result given by Jain and Kanwal [5].

5. Numerical results and discussion

The stress intensity factors (SIF) K_a, K_b, K_c and K_1 given by (4.2)–(4.5) at the edges of the strips and the vertical displacement $|v(x, 0)/v_0|$ near about the rigid strips have been plotted against the dimensionless frequency k_s and distance x, respectively, for three different types of orthotropic materials whose engineering constants have been listed in Table 1.

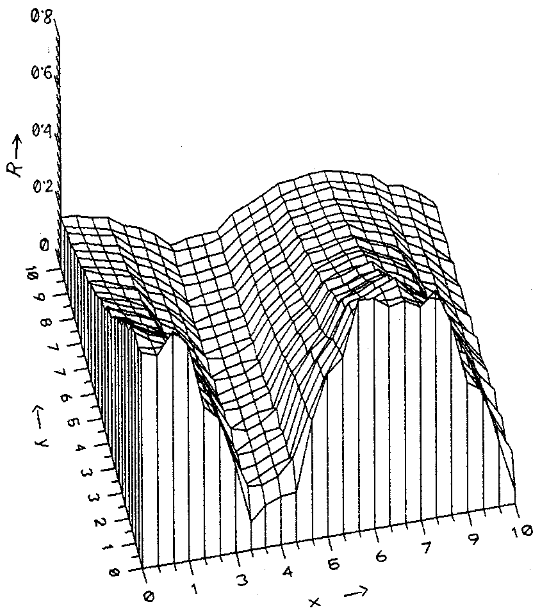

Figure 9. Resultant displacement R of the scattered field vs. x and y. ($k_s = 0{\cdot}1, a = 0{\cdot}2, b = 0{\cdot}4, c = 0{\cdot}6$, Type -I).

Table 1. Engineering elastic constants

E_1 (Pa)	E_2 (Pa)	μ_{12} (Pa)	ν_{12}
Type I Modulite II graphite-epoxi composite:			
$15{\cdot}3\times10^9$	$158{\cdot}0\times10^9$	$5{\cdot}52\times10^9$	0·033
Type IIE-type glass-epoxi composite:			
$9{\cdot}79\times10^9$	$42{\cdot}3\times10^9$	$3{\cdot}66\times10^9$	0·063
Type IIStainless steel-aluminium composite:			
$79{\cdot}76\times10^9$	$85{\cdot}91\times10^9$	$30{\cdot}02\times10^9$	0·31

It is found that, whatever the lengths of the strips, the SIFs at the four edges of the strips increase with increase in the value of $k_s(0{\cdot}1 \leqslant k_s \leqslant 0{\cdot}6)$. From the graphs, it may be noted further that, with a decrease in the length of the inner strip, which might be induced either by increasing a or by decreasing b, the SIF K_a at the innermost edge gradually decreases, whereas the SIFs at the other edges show just the opposite behavior (Figure 2 - Figure 3).
Also, a decrease in the value of the length of the outer strip, which might be induced by increasing the value of c, causes an increase in the values of the SIFs (Figure 4) from which an interesting conclusion might be drawn: i.e., the presence of the inner strip suppresses the SIFs at both edges of the outer strip and the presence of the outer strip suppresses the SIFs at the edges of the inner strip.

The SIF K_a has been plotted (Figure 5) for different orthotropic materials to show the effect of material orthotropy. Similar effects are being seen for other SIFs.

The vertical displacement has been plotted for different strip lengths. It is found from Figure 6-Figure 8 that with the increase in the value of strip length, the displacement increases.

For a fixed material the variation of displacement with frequency is found to be insignificant.

From Equations (2.12) and (2.13) the resultant displacement $R[=\sqrt{(u^2+v^2)}]$ of the scattered field has been calculated for the points (x, y) in the first quadrant and plotted in Figure 9.

From Figure 9 it is evident that the displacement (R) decreases as x and y increase. The nature of the resultant displacement for all the materials are same.

Acknowledgement

We take the opportunity to thank the referees for valuable suggestions for the improvement of this paper. The author S. C. Mandal is grateful to Prof. A. Chakrabarti for continuous encouragements.

References

1. A.K. Mal. Interaction of elastic waves with a Griffith crack. *Int. J. Eng. Sci.* 8 (1970) 763–776.
2. D.L. Jain and R.P. Kanwal, Diffraction of elastic waves by two coplanar Griffith cracks in an infinite elastic medium. *Int. J. Sol. Struct.* 8 (1972) 961–975.
3. S. Itou, Diffraction of an antiplane shear wave by two coplanar Griffith cracks in an infinite elastic medium. *Int. J. Sol. Struct.* 16 (1980) 1147–1153.
4. G.R. Wickham, The forced two dimensional oscillations of a rigid strip in a smooth contact with a semi-infinite elastic solid. *Proc. Cambridge Philosoph. Soc.* 81 (1977) 291–311.
5. D.L. Jain and R.P. Kanwal, Diffraction of elastic waves by two coplanar and parallel rigid strips. *Int. J. Eng. Sci.* 10 (1972) 925–937.
6. S.C. Mandal and M.L. Ghosh, Forced vertical vibration of two rigid strips on a semi-infinite elastic solid. *J. Sound Vibr.* 158 (1992) 169–179.
7. M.K. Kassir and K.K. Bandyopadhyay, Impact response of cracked orthotropic medium. *ASME J. Appl. Mech.* 50 (1983) 630–636.
8. Y. Shindo, H. Nozaki and H. Higaki, Impact response of a finite crack in an orthotropic strip. *Acta Mech.* 62 (1986) 87–104.
9. J. De and B. Patra, Moving Griffith crack in an orthotropic strip. *Int. J. Eng. Sci.* 28 (1990) 809–819.
10. Y. Shindo, H. Higaki and H. Nozaki, Impact response of symmetric edge crack in an orthotropic strip. *JSME Int. J.* 34 (1991) 7–12.
11. S.C. Mandal and M.L. Ghosh, Interaction of elastic waves with a periodic array of coplanar Griffith cracks in an orthotropic elastic medium. *Int. J. Eng. Sci.* 32 (1994) 167–178.
12. T. Kundu and A. Bostrom, Axisymmetric scattering of a plane longitudinal wave by a circular crack in a transversely isotropic solid. *ASME J. Appl. Mech.* 58 (1991) 695–702.
13. K.N. Srivastava and M. Lowengrub, Finite Hilbert transform technique for triple integral equations with trigonometrical kernals. *Proc. R. Soc. Edinburgh* 68 (1970) 309–321.